What is a realistic aversion to risk for real-world individual investors?*

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Abstract

Most frequently used class of utility functions for modelling the investment policy of individual agents by the constant relative risk aversion (CRRA) utility functions. The objective of this paper is to try to provide the plausible risk aversion parameter of individual households under this assumption.

I argue that the risk aversion of an individual investor may be significantly larger than is usually being considered plausible in economic literature. Specifically, I argue that, for the purposes of mediocre risk taking, an average investor's risk aversion parameter is in order of $p = 30$. Few investors, usually with enough experience, exhibit aversion to risk below $p = 20$, while many individuals with little risk-taking experience and distaste for risk endeavors may exhibit $p > 300$. I present empirical evidence for these statements.

The risk aversion coefficients suggested in this paper are an outcome of a partial equilibrium model, analyzing the behavior of real-world individuals. Additionally, we are able to address other issues in financial economics. Namely, the assumption of higher risk aversion resolves the *equity premium puzzle*. It is possible to obtain the *low participation in stock market* even with small transaction costs, assuming high risk aversion for some agents. We can also conclude that *the risk free rate is not too low* for sufficiently high risk aversion.

Introduction

It appears plausible that the usage of CRRA utility function could be a reasonable approximation of the real-world behavior. Unfortunately, it turns out to be somewhat complicated to find an appropriate relative risk aversion parameter. It can be argued that the parameter corresponding to logarithmic utility function ($p = 1$) is appealing since log utility maximizes the rate of growth.

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Soon people started to realize that, in the real world, risk aversion is actually higher than what logarithmic utility implies. Several authors argued that the risk aversion parameter could be around 2. More recently, economists started to consider even higher aversion to risk, finding risk aversion parameter in order of 5 or even 10 to be reasonable. For example, Mehra and Prescott [13] a priori impose an upper bound of 10 for the relative risk aversion parameter p. A notable exception to the general approach is the work of Kandel and Stambaugh [8]. The authors argue that higher risk aversion parameter in the order of 30 might be plausible. They also discuss separation of risk aversion and intertemporal substitution in nonexpected-utility framework.

While major improvements in matching data and attitudes towards risk can be obtained by relaxing the assumption of existence of a utility function, namely the associativity axiom, in this paper I attempt to show that these relaxations are not necessary to satisfactory resolve the microeconomic data. It is possible to make a very restrictive assumption on utility function, while assuming a sufficiently large risk aversion.

I assume that the agent possesses the CRRA (power) utility function in order to make the analysis computationally feasible. This assumption may be arguable and does not hold precisely in reality. However, it is plausible that the usage of CRRA utility functions could serve well as a first order approximation for real-world risk taking behavior. In any case, many authors hold the assumption of CRRA in their models and it is widely accepted.

The assumption of power utility functions can be also understood in the following sense: While an individual agent may not possess exactly the power utility function, we are trying to find the constant relative risk-aversion coefficient, which would "best" approximate the agent's risk-taking behavior. In other words, we strive to answer the problem of what risk aversion coefficient would the agent choose, if one "forces" her to use the power utility function for portfolio choice decisions.

We assume the power (CRRA) utility function of the form:

$$U_p(W_T) = \begin{cases} \frac{W_T^{1-p}-1}{1-p}, & p > 0,\ p \neq 1 \\ \log W_T, & p = 1, \end{cases} \tag{1}$$

where W_T is the investment wealth of agent at time T, as defined in Section 1.

In Section 1 I state the assumptions for portfolio choice or other risk taking behavior of an agent, and define the appropriate reference wealth. In Section 2 I present some empirical evidence from real-world risk taking behavior. In Section 3 I present the implications of the preceding analysis for financial markets, which helps to resolve several major puzzles in economic theory.

1 The model setup

In this paper I assume that individuals make investment decisions based on the level of their personal *investment wealth*. This assumption is probably close to reality. It seems plausible that the theoretically ideal approach should be based on maximizing the utility of lifetime consumption stream, with possible bequest motive. However, it does not appear plausible that an individual investor performs a complicated analysis and difficult estimates of her lifetime consumption. Thus, the assumption of maximizing expected utility of wealth process[1] rather than utility of a consumption stream appears fully compatible with the objective of partial equilibrium analysis, even though it may not be ideal for other purposes. In any case, in Appendix A I show that for power utility there is no difference in agent's optimal investment policy in the two approaches, at least under some reasonable assumptions.

1.1 Agent's investment wealth

A complicated and possibly arguable question is the definition of personal wealth used for making the portfolio choice decisions (further referred to as *investment wealth*). Under the somewhat unrealistic assumptions of frictionless market (namely unlimited borrowing at risk free rate) and non-stochastic labor income, it is clear that the present value of labor income (the human wealth) can be added to the definition of agent's investment wealth. Even though these assumptions are not fully realistic, it appears plausible that a significant portion of, or some form of certainty equivalent of personal wealth can be considered as part of agent's investment wealth.[2] Alternatively, the reader can assume that the analysis is done from the point of view of an agent close to retirement, where most of the human capital is already exhausted.

Note that in this setting, with properly discounted human capital being considered to be a part of agent's investment wealth, it may not be necessarily the case that risk aversion increases with age of the agent. In fact, it could easily happen that the agent's risk aversion is constant or theoretically even decreasing over life time.

I argued that a significant portion of human wealth (or labor income) should be part of the definition of investment wealth. It appears plausible that for purposes of calculation of the investment wealth one should subtract the present value of some minimal required consumption. For example, paying a mortgage is a necessary consumption, which is paid by labor income.

[1] A useful property of power utility functions is that the optimal policy is, under mild technical conditions on the underlying stochastic process, *myopic*, i.e., it does not depend on the somewhat arbitrary time horizon. For discrete time gambles it is sufficient to assume independence of the games. No assumption is needed for logarithmic utility function.

[2] I abstract from the complicated option-type analysis of very young agents, with possibly large human wealth, but with no cash-on-hand. While these agents would like to participate in risky endeavors, they may be forced to temporarily stay out of the market due to borrowing constraints.

The net result of the suggested approach is that only a certain portion of human wealth is added to agent's investment wealth. If the agent pays for house in cash rather than through mortgage, and thus the currently available cash-on-hand is significantly smaller, a larger portion of human wealth would be added to the investment wealth. Consequently, the investment wealth is approximately the same, regardless of agent's housing policy.

In this setup, the assumption of CRRA utility function, namely infinite marginal utility for zero wealth, appears rather plausible compared to other models. It is important to stress that, in the setup of this paper, losing the entire investment wealth means more than just bankruptcy. With some exaggeration, one could say that should the agent lose her entire wealth, the result would be life on social welfare. (Or more precisely, life with minimal consumption but no more than that.) Thus, it is more than just a possibly short or mediocre term inconvenience as is in the case of losing monthly or yearly income. The risk aversion coefficient is related to the *entire* investment wealth.

2 Plausible risk taking behavior of human beings

For the following considerations it is important to keep in mind the setup of the model: The portfolio choice is taken as a proportion of the current investment wealth, where the investment wealth includes appropriately discounted, some kind of certainty equivalent of, labor income. My arguments would not hold if an agent calculates her investment as a proportion of currently available cash, while she is reasonably certain of (and ignores for the purposes of investing) the next years' substantial income. In such a case, losing 95% of wealth defined as currently available cash might be no more than temporary inconvenience.

Perhaps even more understandable could be for the reader to assume that the analysis is done for a reasonably wealthy individual, whose remaining labor income exhibits very low-risk or who is close to retirement. If the agent is close to retirement, the investment wealth will be close to the actual wealth level accumulated by the agent over life time.

A simple analysis of draw-down probabilities (see Appendix B) shows that the logarithmic utility is extremely risky and utterly implausible In fact, the draw-down probability (11) suggests that any risk aversion $p \leq 3$ is rather implausible – for example, the probability of psychologically hardly conceivable draw-down of 50% or more is still non-negligible 3.125% for $p = 3$.

While the draw-down probability provides a lower bound on the realistic risk aversion coefficients, it does not provide an upper bound. The psychological inconvenience of experiencing losses may be caused by more factors than just long-term wealth fluctuations. Even though the draw-down probability may not invalidate the assumption of any risk aversion parameter higher than 5, in the remainder of this section I will argue that, *for the purposes of taking* *investment risks*, it is reasonable to assume $p > 10$ even for investors experienced in risk taking, and significantly larger p in order of 30 or more for standard households.

In this context it is important to note that the considered risk aversion coefficients are designed to match the realistic aversion to risk for relatively small risks and low expected return, with not too skewed distribution of the results. Taking this kind of risk corresponds to investing in financial markets. Possibly different (and significantly lower) risk aversion may be observed if a subject must choose between two bad scenarios – a choice between a certain loss versus a chance of no loss with (most likely) a little higher loss. This kind of examples demonstrates the well known *Allais paradox* [1] to existence of a utility function. Similarly, different (and significantly lower) risk aversion would be observed if a proposed gamble includes a loss of relatively large sum vs. a win of arbitrarily high (infinite) sum. In [10] the author shows examples of these types of gambles and concludes that people do not behave according to Savage-type of utility functions.

A related point to stress is that we assume that the agent cannot choose a random draw out of long-run distribution of wealth (for example, after 10 years), and 'close eyes'. The agent must repeatedly take small risks with small expected returns, and experience corresponding losses.

2.1 Risk aversion of professional gamblers

Empirical evidence can be found by analyzing the behavior of *professional gamblers.* As a real world example consider the professional or semi-professional *blackjack players.* In this casino game a knowledgeable player can reach positive expected value by the technique of *card counting.* As a recognized author of simulation/analysis software, which is used and well-known in the industry (see for example www.sba21.com), I had the opportunity to make a pool among professional and semi-professional blackjack players, while participating in a private Internet forum. While the data sample of the pool is not large, I can make the following conclusion:
In general, long-term professional gamblers exhibit risk aversion $p \geq 40$. Many professional gamblers are in the area of p close to 100.[3] It is very exceptional to find players with risk aversion lower than $p = 10$. Individual risk aversion lower

[3] Most professional players use *half Kelly* to *third Kelly* betting strategy. Full Kelly, see [9], is equivalent to logarithmic utility. By using *half Kelly* the player implies that his bet sizes are half of what Kelly (logarithmic) utility would suggest, which practically exactly corresponds to CRRA with $p = 2$. However, it is important to realize that the reference point is not player's investment wealth, rather some artificially defined *blackjack bankroll.* The bankroll of a professional player is usually less than 5% of wealth, in which case half Kelly betting corresponds to $p \geq 40$. Most players define the bankroll and afterwards "flat bet", i.e., do not scale the bet sizes as a function of current bankroll level. The appropriate *Kelly fraction* (usually half Kelly to third Kelly) is chosen so that the probability of *risk-of-ruin* is small, which is convenient from psychological point of view. In case of the unfortunate loss the player would replenish the bankroll, possibly with a little lower bet sizes.

than $p = 10$ could possibly be found amongst rather new gamblers with relatively very little wealth who are forced to exhibit low risk aversion, and suffer significant fluctuations in their wealth, due to practical constraints (minimal gaming volume, fixed costs for travelling, etc.) However, even these individuals usually increase their aversion to risk over time, as they start accumulating sufficient wealth.

It is plausible that a professional gambler, used to and having much experience in risk taking and with sufficient experience, will exhibit low risk aversion compared to a standard household. Also, it appears plausible to assume that a professional gambler is no more risk averse compared to an average investor. The related issue of ambiguity of the size of stock drift, compared to precisely known odds in a card game, is yet another reason why the equilibrium risk aversion observed in financial markets should be greater than risk aversion of professional gamblers.

2.2 Estimating personal risk aversion

The only known attempt to measure a real risk aversion was performed by Binswanger [3] in rural India. The author offered a positive games to randomly chosen farmers in India. The farmers could decide between accepting certain monetary sum, or participating in a gamble with better expected return but risk. This is a very good experiment measuring the real-world aversion to risk, as the participants deal with real and significant monetary sums.

Binswanger obtains a reasonable size of relative risk aversion under the assumption of a utility function with constant partial risk aversion in the neighborhood of the payoff levels. Except for the smallest game (only 0.50 Rs certain payoff), we get very similar numbers when assuming constant relative risk aversion. Taking the suggested wealth level of 10,000 as given, the experiment suggests that the relative risk aversion of individual farmers under the assumption of power utility is in the range of 10-30.

The obtained relative risk aversion of 10 to 30 is somewhat lower (but not too much lower) than I suggest in this paper. However, one should consider the fact that the suggested wealth of 10,000 is probably lower than the investment wealth as defined in this paper. More importantly, the performed experiment is psychologically very different in comparison to *taking* investment risks. The *Allais paradox* implies that the risk aversion towards taking investment risks (and suffering unavoidable losses) will be significantly higher than risk aversion towards only up-side games. Unfortunately, it appears practically impossible to conduct an experiment in which the participants would have to pay significant sums in case of bad luck. A possible way to try to estimate the realistic aversion for win/loss propositions is to look at the behavior of professional gamblers, as outlined in the previous section.

The formal definition of bankroll is especially useful for blackjack teams. Members of a blackjack team form a joint bankroll, decide on Kelly fraction, and then share risk and returns of gaming.

In order to gain some personal intuition into win/loss risk taking behavior it may be useful to imagine a sequence of not too large gambles with positive expected return, and consider a personal attitude towards such gambles. The absence of ambiguity implies that, for rational investors, one could hardly expect higher risk aversion in gaming examples compared to the stock market.

For simplicity I will assume discrete-time gambles. The analysis is completely analogical for continuous time investment and risk-taking behavior.

We define *certainty equivalent* CE of a random variable (a gamble) X as

$$CE(X) = U^{-1}\big(\mathrm{E}\, U(W + X)\big), \tag{2}$$

where U is the agent's utility function, and W is agent's investment wealth just before the agents makes the decision whether to participate in gamble X or not. The agent will participate in the gamble if and only if

$$CE(X) > 1.$$

This follows immediately by observing that $CE(0) = 1$.

After substitution we immediately obtain for CRRA utility functions

$$CE_p(X) = \begin{cases} \left(\mathrm{E}\left[(W+X)^{1-p}\right]\right)^{\frac{1}{1-p}}, & p > 0, p \neq 1, \\ \exp\left\{\mathrm{E}\log(W+X)\right\}, & p = 1. \end{cases}$$

An agent with power utility with parameter p will participate in the gamble if and only if $\mathrm{E}\left[(W+X)^{1-p}\right] < 1$ for $p > 1$, if and only if $\mathrm{E}\left[(W+X)^{1-p}\right] > 1$ for $p < 1$, and if and only if $\mathrm{E}\log(W+X) > 0$ for $p = 1$.

For simplicity assume that the agent is close to retirement and her accumulated wealth equals $W = \$200,000$. This level of wealth is clearly very conservative. The average wealth of a person in developed countries will probably be significantly higher. Assuming higher wealth would make all the arguments even stronger by implying higher risk aversion of an average person.

Suppose that an agent with investment wealth of \$200,000 faces a gamble of losing \$100 with probability $\frac{1}{2}$, and winning \$120 with probability $\frac{1}{2}$. The expected return of this gamble is \$10, which is 10% of the risked amount. Will the agent take part in this gamble?

Now consider the same gamble, this time with 10 times higher bet size — losing \$1,000 versus winning \$1,200. Last, we can do the same consideration for large amounts of \$10,000 vs. \$12,000.

The question is, what minimum risk aversion is necessary in order to refuse the gamble. I argue that hardly any moderately wealthy individual would take the large gamble of \$10,000 vs. \$12,000. Most people will probably not take the gamble of \$1,000 vs. \$1,200 either, although some people used to risk taking may be willing to go for the 10% expected return and risk \$1,000. many people may take the \$100 vs. \$120 gamble. Still, I would argue that it is not that rare to find individuals who would refuse to risk \$100 for making extra \$10 in expected value.[4]

The corresponding risk aversion parameters for CRRA utility functions are the following: A person with $p \leq 3.33$ would take the large gamble. In other words, a risk aversion of more than 3.33 is required in order for the agent to refuse the large gamble. For the medium gamble, the borderline p equals 33.3. If one were to assume that a plausible risk aversion parameter is less than 10, one would have to conclude that moderately wealthy person would happily risk \$1000 for a 10% expected profit. (In fact, a a person with $p = 10$ would gamble up to \$3,300 for the 10% expected return, see Figure 2.2.) Last, in order to refuse the gamble with \$100 (which can realistically happen), the person's risk aversion parameter must be larger than 330!

Another interesting question is how large bet size would an individual optimally choose. Suppose that an agent can choose the size of the bet, losing with probability 50% and winning 1.2 times the bet size with probability 50% as before. It can be shown that the optimal bet size is approximately half of the borderline bet size, for which certainty equivalent equals 1.[5]

Finding the optimal bet size is equivalent to maximizing certainty equivalent (2). Is is easy to show that the optimal bet size b_p for this particular game equals

$$b_p = \frac{1.2^{\frac{1}{p}} - 1}{1.2^{\frac{1}{p}} + 1.2} W.$$

(See Figure 2.2.)

For example, if the agent possesses power utility with $p = 5$, she would like to bet \$3,220 for expected profit of \$322. The optimal gamble is still rather implausible \$1,659 for $p = 10$. For more plausible risk aversion parameters of $p = 20$ and $p = 30$ the optimal bet sizes are \$829 and \$550 respectively. Individuals which are highly averse to taking any risks and optimally would like to bet just \$83, exhibit risk aversion with $p = 200$.

I would argue that the proposed game has quite reasonable parameters, and is hardly subject to much confusion or misinterpretation or psychological biases. The probability of win or loss are the same (50%), and the corresponding edge of 10% is not extremely large nor extremely small.

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[4]Many individuals may be resistant against taking one single gamble, while they would be willing to take a sequence of identical gambles at once. This kind of behavior is related to the non-existence of a utility function. Note, however, that a yearly investment in a stock market roughly corresponds to playing only 14 such games per year, after which the standard deviation of returns is 3 times larger than the expected value.

[5]The optimal investment proportion is exactly half of the borderline bet size under the assumption of continuous time and geometric Brownian motion for the risky investment (say stock).

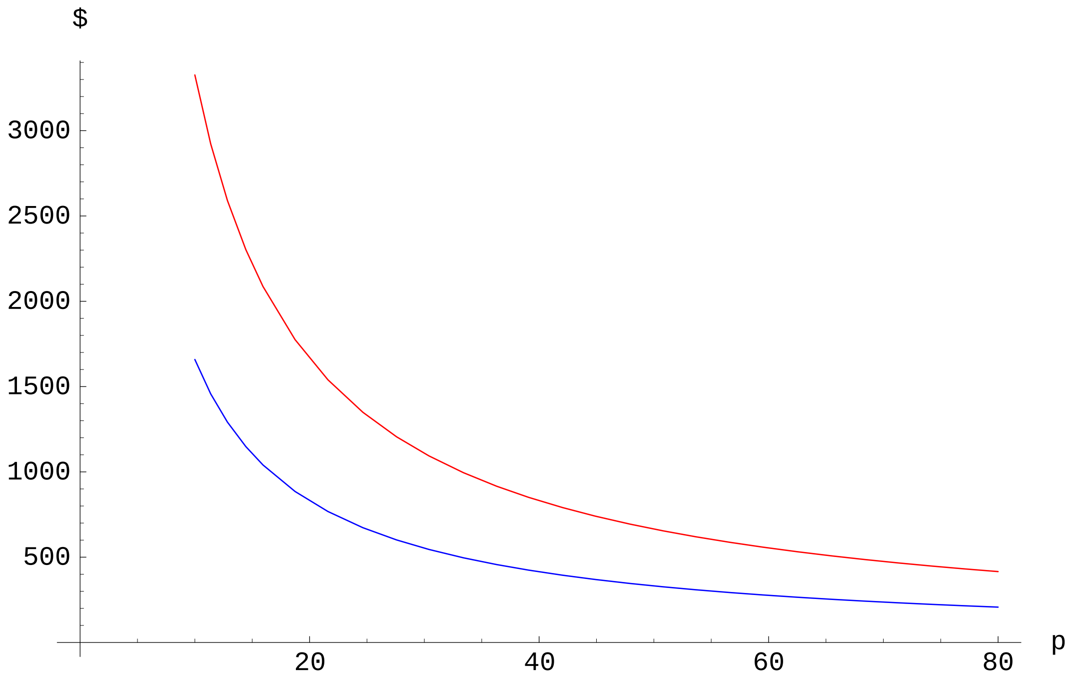

fig. 1: The optimal (blue) and maximum (red) bet size for $200,000 inv. wealth.

2.3 Investments in financial markets

2.3.1 Sophisticated investors investing in aggressive hedge funds

Various kinds of hedge funds tend to exploit market inefficiencies, usually in some rather sophisticated ways. Different groups of hedge funds are from a large part independent in their risk/return profiles. Examples include hedge funds of type *Managed Futures* (exploiting trends and other technical analysis), *Macro Discretionary* (analyzing macroeconomic trends), *Long/Short Equities* (purchasing presumably undervalued stocks and hedging those positions by shorting other stocks in similar market segments, so that the fund performance is largely independent of overall stock index returns), hedge funds exploiting the *market price-of-risk* (for example, funds speculating on relative appreciation of currencies with higher interest rate, or funds exploiting the risk premium in the forward curve.)

While investments in hedge funds require non-trivial information and presumably rather deep market experience, many wealthy individual investors indeed seek these types of investments. A hedge fund with Sharpe ratio of 1.2 is probably not an exception. A realistic investment strategy of an aggressive hedge fund corresponds to power utility with $p = 3$, no less than $p = 2$.[6] Note

[6]By "aggressive hedge fund" I mean a risky endeavors with expected yearly returns of several tens of percent. This was indeed the standard type of a hedge fund, living out of fees based on realized returns, until the unfortunate LTCM event. It seems that at present many so-called hedge funds are rather low-risk investments with little excess returns, living of fixed yearly percentage

that even an aggressive hedge fund would hardly intentionally use risk aversion of $p < 2$. Such a policy would be rather irrational from the fund manager's point of view, since for $p < 2$ the probability of draw-down of 50% of funds equity would become quite large. (See Theorem 2, although the actual risks will be even larger than the theorem suggests due to non-normality of log returns and the existence of 'heavy tails'.) It is usually assumed in the industry that a draw-down of more than 50% of equity will imply a liquidation of the fund.

Assuming a correctly evaluated (out-of-sample) Sharpe ratio of 1.2 of a hedge fund with investment strategy corresponding to $p = 3$, it can be shown that, under some technical assumptions, the expected yearly excess return of such hedge fund approximately amounts to 40% p.a. A plausible individual's investment in this kind of speculation may be around 10% of personal investment wealth, corresponding to agent's risk aversion parameter of $p = 30$.

Observe that even if an investor has 50% or more of his wealth invested in risky assets (hedge funds), this does not necessarily imply a low risk aversion coefficient. It is enough to find 5 different hedge funds with independent returns, and invest 10% of investment wealth in each of them. (For simplicity assume similar risk/return profile of each fund.) In this case, an investment of 50% in 5 hedge funds, each with investment strategy $p = 3$, still corresponds to risk aversion of $p = 30$ of the investor.[7]

2.3.2 Investments in standard stock market

I would argue that the direct or indirect investment proportions into standard "buy-and-hold" stock portfolio hardly exceeds 10% of investment wealth of most individual agents. More sophisticated investors with higher proportion of their wealth invested into risky assets are investors with individual investment opportunities, for example investors investing in various kinds of more or less independent hedge funds, as discussed in Section 2.3.1.

A standard household without access to rather sophisticated investment opportunities may have for example 30% of investment wealth invested in various mutual funds. However, only a small portion of these investments gets indirectly allocated into standard stock market, while most of the funds are invested with little risk (and little excess return).

Suppose that a standard portfolio of stocks exhibits instantaneous excess

commissions. These kinds of endeavors tend to approach, in some sense, the style of standard mutual funds, relatively low risk investments. I do not consider these for purposes of this analysis.

[7] The optimal investment policy in additional *independent* investment opportunity is not influenced by other existing or expected investments. This is precisely true in continuous time, with the possibility of continuous re-balancing of portfolio. If the re-balancing is not done continuously, rather "as needed" (as soon as the position is too far from optimum), any difference is negligible. If re-balancing of portfolio can be done very rarely, say once a year, the optimal investments in each endeavor will be a little lower. The difference will still be rather negligible if each individual investment is only a small portion of wealth (say 10% as in our example), and the number of investments is not too large. The effect of re-balancing would become significant if the total risky investments shall exceed 100% of agent's investment wealth.

return of 6% p.a.[8] with instantaneous volatility of 18% p.a. An investment of 10% of wealth into this stock portfolio corresponds to risk aversion parameter $p = 18.52$. As argued above, it is reasonable to conclude that few individuals would allocate whole 10% of wealth into a stock index, just assuming 6% of excess return with corresponding risk, and performing the simple buy-and-hold strategy. Larger investment allocations are usually based on more optimistic (and often unjustifiable) assumptions about the underlying risk/return profile, overconfidence in personal information, and similar.[9]

3 Resolution of several economics puzzles

3.1 The equity premium puzzle

In the preceding sections I argued that an investor's risk aversion is usually more than $p = 30$. In particular, it appears that there is *no equity premium puzzle*, as defined in [13]. The equity premium puzzle disappears as soon as one assumes sufficiently high risk-aversion coefficients in the order of $p = 30$.

3.2 Low participation rate in the stock market

One of the puzzles of modern economics is the observed low participation of households in the stock market. The standard argument is that with finite relative risk aversion and positive excess return of a stock market, each household with positive investment wealth should hold some positive proportion of wealth in stocks.

The explanation of this puzzle appears not to be too complicated by considering higher risk aversion coefficients and modest transaction costs for participation in the stock market. Through the gaming example I argued that many individuals with little exposure to risk in general, and little understanding of risky investments, may indeed exhibit very large aversion to risk, corresponding for example to risk aversion parameter $p = 200$ or even larger.

I will analyze the certainty equivalent loss of no participation in the stock market given a risk aversion parameter p. For technical simplicity assume the geometric Brownian motion for the underlying stock price. The evolution of agent's wealth is expressed by (12). We can calculate the certainty equivalent:

$$CE_p = U_p^{-1}\big(\mathrm{E}\left[U_p(W_T)\right]\big) = \left(\mathrm{E}\left[W_T^{1-p}\right]\right)^{\frac{1}{1-p}} = \exp\left\{\frac{1}{2p}\frac{\alpha^2}{\sigma^2}T\right\}.$$

In particular, for $p = 100$ (not that large and arguably not an exceptional risk aversion), $\alpha = 6\%$, $\sigma = 18\%$, and $T = 1$ we obtain $CE = 1.000556$. Thus,

[8] The argument also holds when assuming lower excess return.

[9] It can certainly happen that a misled individual with unrealistic assumptions about a particular stock allocates an unjustifiably high proportion of his financial wealth into this stock. However, such an event does not imply low risk aversion coefficient, since the investor erroneously assumes enormous excess return based on his personal information.

by ignoring the stock market the agent loses, in certainty equivalent, the yield of 0.0556% of his investment wealth per year. This loss (equal to $111 per year for agent with investment wealth of $200,000) is low indeed, and small transaction costs such as setting up a brokerage account, or even just thinking and learning about the stock market, imply that it is fully rational for highly but probably not exceptionally risk averse agent to stay out of the market.

3.3 The low risk free rate is not too low

A general consumption based model (9) implies for the dynamics of risk free rate r_t (see for example [5, pp. 32])

$$r_t\,dt = \beta\,dt - c_t \frac{u''(c_t)}{u'(c_t)}\,\mathrm{E}_t\left[\frac{dc_t}{c_t}\right] - \frac{1}{2}c_t^2 \frac{u'''(c_t)}{u'(c_t)}\,\mathrm{E}_t\left[\frac{d\langle c_t\rangle}{c_t^2}\right]. \tag{3}$$

Equation (3) gives the *shadow price* of individual consumer's risk free interest rate. If the market risk free rate was higher the consumer would adjust her position by consuming less and investing more in a risk free asset. If the market risk free rate was lower the consumer would (in a frictionless market) borrow at risk free rate and consume more.

Assume now the power utility functions U_p. For simplicity of exposition, we further assume a geometric Brownian motion for the consumption process

$$dc_t = c_t\big(\alpha\,dt + \sigma\,dB_t\big).$$

Equation (3) for the risk free rate r then simplifies to

$$r = \beta + \alpha \cdot p - \frac{1}{2}\sigma^2 \cdot p(p+1). \tag{4}$$

Formulas (3) and (4) show that consumption growth is high when interest rates are high. Since higher coefficient of risk aversion p makes interest rates more sensitive to consumption growth, an argument is often made that higher risk aversion implies higher interest rate, and that a high risk aversion coefficient cannot explain the low risk free rate observed in the economy[10]. People would often ignore the *precautionary savings* effect expressed by $\frac{1}{2}\sigma^2 \cdot p(p+1)$ since the observed volatility of consumption is low (say in the order of 3%). See Kandel and Stambaugh [8] for analysis of the precautionary savings effect.

The size of risk free rate is a quadratic function of p. For sufficiently high p, the effect of precautionary savings starts to dominate the effect of consumption growth. We will see that the interest rate puzzle is satisfactory explained when considering sufficiently high (and plausible) risk aversion coefficients, and

[10] One of the weak points of this argument is the implicit assumption of frictionless market, where people can borrow at risk free rate. However, it is true that the same conclusion holds even for high bid/ask spread for borrowing and lending

when confronting the model with real world consumption growth and volatility data.[11]

In this context it is interesting to note the results in Weil [14]. The author attempts to resolve the equity premium puzzle and risk free rate puzzle with a class of more general Kreps-Porteus non-expected utility framework with independent parametrization of attitudes toward risk and attitudes toward intertemporal substitution. In his model, the intertemporal elasticity of substitution is not required to be the inverse of the constant coefficient of risk aversion, as is the case for power utility functions. He concludes that, despite the extra degree of freedom, the model cannot explain the economics puzzles, and argues that the existence of heterogeneity and of serious market imperfections is necessary in order to attempt to explain the economy. The author claims that *"Under the expected time-additive utility restriction $\rho = \gamma$, decreasing the intertemporal elasticity of substitution [inverse of ρ] amounts to increasing the coefficient of relative risk aversion [γ], and results in the simultaneous rise of the risk premium and the risk-free rate."*

I claim that this statement is not correct, as shown by Equation 4. (The results are practically the same regardless of continuous time or discrete time modelling.) The interest rate is an increasing function of risk aversion for $p \leq \alpha/\sigma^2 - \frac{1}{2}$, while it is a decreasing function of risk aversion for greater p, as can be easily seen by taking the derivative of (4). In Table 1 we can see for different value of expected growth rate α and volatility σ, the threshold size of p, for which the risk free is already low (equals β). The size of p that maximizes the risk free rate is exactly one half of the values in the table.

[14] further claims: *"There is no way to fit both the level of the risk-free rate and the risk premium when the VNM [Von Neumann-Morgenstern utility] restrictions is imposed."* I show next that this conclusion is not necessarily correct either, since it is enough to consider high but realistic risk aversion of agents.[12]

From Figure 1 we can see that no "low risk free rate puzzle" is present for expected consumption growth of 3% and volatility of consumption of 2%, with high risk aversion coefficient close to $p = 150$. For example, if the representative agent's risk aversion is $p = 149$, we get $r = \beta$. For $p = 150$ we already have $r = \beta - 3\%$. For higher standard deviation of consumption of 2.5%, Table 1 shows that the "borderline" risk aversion is 95, for which $r = \beta$. In [13] the authors use the historical average consumption growth of 1.83% with standard

[11] The precautionary savings effect is not an artifact of power utility functions. The effect with the same order exists for, for example, exponential utility with constant absolute risk aversion. Intuition suggests that any reasonable utility function should exhibit the precautionary savings behavior (negative third derivative of utility function), as with more volatile consumption people would tend to save more since they are worried about low consumption states. With extra motivation for savings the shadow interest rate must decrease.

[12] In [14] the author argues that a realistic coefficient of relative risk aversion must be in the range of 1 to 5, probably close to 1. A simple argument of draw-down probabilities from Theorem 2 shows that "risk aversion close to 1" is an utterly implausible assumption.

deviation of 3.57%. In this case, the "borderline" risk aversion is just $p = 27.7$. However, modern data indicate that the actual volatility of consumption is lower.

$\alpha \backslash \sigma$	1.50	1.75	2.00	2.25	2.50	2.75	3.00	3.25	3.50	3.75	4.00
1.50	132	97	74	58	47	39	32	27	23	20	18
1.75	155	113	87	68	55	45	38	32	28	24	21
2.00	177	130	99	78	63	52	43	37	32	27	24
2.25	199	146	112	88	71	59	49	42	36	31	27
2.50	221	162	124	98	79	65	55	46	40	35	30
2.75	243	179	137	108	87	72	60	51	44	38	33
3.00	266	195	149	118	95	78	66	56	48	42	37
3.25	288	211	162	127	103	85	71	61	52	45	40
3.50	310	228	174	137	111	92	77	65	56	49	43
3.75	332	244	187	147	119	98	82	70	60	52	46
4.00	355	260	199	157	127	105	88	75	64	56	49

Table 1: Threshold p for which r equals to β, for various % values of α and σ.

To understand why the high risk aversion coefficient for explaining the consumption data (and consequently the size of risk free rate) is plausible, it is important to note that the risk aversion related to consumption growth and volatility of consumption is influenced by each consumer in the economy. Thus, the risk aversion should be significantly larger than the risk aversion of investors in financial markets, who are a special selected group. It is plausible that, for the purposes of risk free rate analysis, the risk aversion of a representative agent may easily exceed $p = 150$. (The gaming example shows that the *median* risk aversion could well be above $p = 300$.)

An argument could be made that with high coefficients of risk aversion, the size of interest rate, as a quadratic function of p, is very sensitive to small changes in p. Thus, other things being equal, a small change in consumers' attitude towards risk causes a large change in interest rate. For example, with $\alpha = 3\%$, $\sigma = 2\%$, $\beta = 5\%$, the size of shadow risk free rate for $p = 140$ is 30.2%, while it is -30.2% for $p = 160$. Similarly small changes in volatility would, other things being equal, cause enormous changes in the risk free rate, which does not move that much in reality.

The problem of this argument is the assumption of *other things being equal*. The point is that small changes in attitude towards risk should, at the first place, cause *small* changes in the volatility of consumption, possibly also changes in the size of consumption growth. Similarly, volatility of consumption may be a result of equilibrium, where higher risk aversion implies lower volatility.

4 Conclusion

In this paper I have questioned the standard assumption in economics concerning individual agent's risk aversion coefficient. I first argue that it is plausible to assume that investment behavior is based on agent's properly defined personal investment wealth, rather than on rather complicated analysis of agent's lifetime consumption, although both approaches yield similar investment policy. In this setup I provide empirical evidence that the coefficient of relative risk aversion for *professional gamblers*, i.e. group of individuals with presumably one of the lowest aversion to risk amongst human population, is usually larger than $p = 40$. By providing a gaming example I attempt to show that a realistic risk aversion parameter is larger than $p = 30$, hardly $p < 20$ for any standard household. Furthermore, it is probably no exception to find highly risk averse individuals with corresponding risk aversion coefficient of $p = 200$ and more.

Assuming high risk aversion of individual households resolves several major puzzles in economic theory. Namely, the *equity premium puzzle* is resolved. Very small transaction costs imply the observed *low participation rate in the stock market*. For a highly risk averse agent, the certainty equivalent loss of no participation in the stock market is less than 0.056% per year. Furthermore, the *low risk free rate* is not necessarily a problem, even under frictionless market assumption.

An interesting question remains why economists in general believe that a plausible risk aversion parameter should be low, starting historically at utterly implausible values between 1 and 2, and nowadays still arguing that plausible risk aversion implies $p \leq 10$. Only very few papers argue that a risk aversion coefficient higher than $p = 30$ is realistic, see for example [4], [7]. This is despite the fact that a sufficiently high coefficient of relative risk aversion helps to better explain the observed economic data.

I conjecture that one of the reasons for the usual assumption of arguably too low risk aversion of individual investors may be caused by a large sensitivity of this type of analysis to standard model assumptions. Namely, the standard Lucas [12] model assumes a complete perishability of the dividend process. Consequently, the aggregate dividend D_t must equal the aggregate consumption C_t in each period, while storage and "borrowing" is precluded. This may be a crucial assumption since it implies that a loss in an "aggregate gamble", say sudden drop in the stock market, forces agents to drop consumption significantly. In reality, smoothing consumption is at least partially feasible.

For the purposes of risk aversion analysis a similar consideration as above should be done for an individual agent. If an agent loses in a gamble it does not imply that she needs to drop consumption significantly in the current period. Making this implausible assumption is rather critical as it could easily imply very low risk aversion coefficients of an average person, just by showing that individuals are willing to participate in small gambles. In reality, a loss in a not too large gamble will have rather negligible effect on current consumption since the loss will be spread over lifetime consumption. In Section 2.2 I argue that, for a modestly wealthy individual, the willingness to accept \$1,000 gamble with expected return 10% does not imply low risk aversion, the implication is just $p \leq 33.3$. On the other hand, refusing such a gamble implies risk aversion $p \geq 33.3$. Willingness to participate in a similar gamble with bet size \$100 implies nothing more than $p \leq 330$.

A Utility of terminal wealth versus optimal consumption

Following is a well-known result, which I would like to stress for purposes of this model: For CRRA utility functions and under reasonable assumptions, the agent's optimal (instantaneous) investment in risky asset is the same when maximizing the utility of terminal wealth and the utility of lifetime consumption stream.[13] This shows that the results and arguments of this paper concerning realistic risk aversion are robust and do not depend on particular technical setup of the model, namely maximization of utility of wealth.

In both models below the agent starts with initial wealth W_0. In the first case, the agent maximizes

$$\mathrm{E}\left[U_p(W_T)\right] \tag{5}$$

for some arbitrary $T > 0$. In the second case the agent maximizes utility of lifetime consumption

$$\mathrm{E}\int_0^\infty \mathrm{e}^{-\beta t}\, U_p(c_t)\, dt. \tag{6}$$

Trading and consumption are subject to standard admissibility constraints.

Theorem 1. *Assume that the stock price follows a geometric Brownian motion with excess return $\alpha - r$, where r is the risk free rate:*

$$dS_t = \alpha S_t\, dt + \sigma S_t\, dB_t.$$

Then, when maximizing utility of terminal wealth (5) or utility of lifetime consumption (6), the agent's optimal investment policy is instantaneously the same. Namely, the agent invests the proportion

$$\pi_t = \frac{1}{p}\frac{\alpha - r}{\sigma^2}, \qquad \textit{for all } t \in [0, T], \tag{7}$$

of his wealth into the risky asset.

[13] Similar result was probably first shown by Hakansson in [6] in discrete time setting.

A.1 Maximization of expected utility of terminal wealth

The agent's wealth evolves as

$$dW_t = (\alpha - r)\pi_t W_t \, dt + rW_t \, dt + \sigma \pi_t W_t \, dB_t,$$

where π_t is the agent's proportion of wealth invested into risky asset. π_t satisfies the admissibility constraints: It is a stochastic process predictable with respect to $\sigma(B_t, t \geq 0)$, and it is appropriately restricted to avoid doubling strategies and similar. After solving the SDE above we can write

$$W_T = W_0 \exp\left\{ \int_0^T \Big((\alpha - r)\pi_t + r - \frac{1}{2}\sigma^2 \pi_t^2 \Big) \, dt + \sigma \int_0^T \pi_t \, dB_t \right\}.$$

We obtain

$$\mathrm{E}\left[\frac{W_T^{1-p} - 1}{1-p} \right] = \frac{W_0^{1-p} \exp\left\{ (1-p) \int_0^T \big((\alpha - r)\pi_t + r - \frac{1}{2} p \sigma^2 \pi_t^2 \big) \, dt \right\} - 1}{1-p}. \tag{8}$$

(We can take the expectation since π_t satisfies the admissability constraints.)

Maximizing (8) over π_t yields Formula (7).

A.2 Maximization of value function in a consumption based model

The agent maximizes the value function

$$v(W_0) = \sup_{C, \Pi \in \mathcal{A}(W_0)} \mathrm{E} \int_0^\infty \mathrm{e}^{-\beta t} \, U_p(c_t) \, dt \tag{9}$$

for admissible controls $C = \{c_t, t \geq 0\}$ (consumption process) and $\Pi = \{\pi_t, t \geq 0\}$ (the process of proportion of wealth invested in stock).

The HJB equation corresponding to the value function is

$$\sup_{c_t, \pi_t} \left\{ U_p(c_t) - c_t v'(x) - \beta v(x) + \big((\alpha - r)\pi_t x + rx \big) v'(x) + \frac{1}{2} \sigma^2 \pi_t^2 x^2 v''(x) \right\} = 0.$$

The optimal controls become

$$c_t = I\big(v'(x)\big) = \big(v'(x)\big)^{-\frac{1}{p}}, \qquad \pi_t = -\frac{\alpha - r}{\sigma^2} \cdot \frac{v'(x)}{x \, v''(x)}.$$

The solution to the HJB equation is given by

$$v(x) = \frac{A^{-p}(p) \, x^{1-p} - \beta^{-1}}{1-p}, \qquad \text{where } A(p) = \frac{\beta - r(1-p)}{p} - \frac{1}{2} \frac{(\alpha - r)^2}{\sigma^2} \cdot \frac{1-p}{p^2}.$$

We immediately obtain

$$c_t = A(p) \cdot x_t, \qquad \pi_t = \frac{1}{p} \frac{\alpha - r}{\sigma^2}, \quad t \in [0, \infty). \tag{10}$$

Thus, the proportion π invested in risky asset is also given by (7).

B Calculating draw-down probability

Theorem 2 (Draw-down probability). *Let the stock price follow the geometric Brownian motion with arbitrary drift $\alpha \neq r$, and arbitrary positive variance. Let the investor posses the power utility function U_p for some $p \geq \frac{1}{2}$. Let $P_p(x)$ denote the probability that the investor's discounted wealth will ever fall below fraction x of the initial wealth, $0 \leq x \leq 1$. (We refer to this situation as relative draw-down of size $1 - x$.) Then*

$$\boxed{P_p(x) = x^{2p-1}} \tag{11}$$

Corollary. *For logarithmic utility agent ($p = 1$), the probability of experiencing a draw-down more than $1-x$ percent of the current investment wealth is $P_1(1-x) = 1 - x$. More specifically, the agent will be, at some point in the future, losing more than 50% of her entire initial wealth with probability $P_1(0.5) =$ 50%. The probability of ever being down more than 90% of the initial wealth is $p_1(0.1) = 10\%$.*

In retrospection, by considering just the draw-down probabilities it is clear that no real-world individual could exhibit as small risk aversion as implied by logarithmic utility. The fluctuation of wealth would be enormous, as the agent would repeatedly experience draw-downs in her entire wealth in the order of 90%, 95%, or even 99%.

Remark 1. Formula (11) is valid for $p \geq \frac{1}{2}$. For $p = \frac{1}{2}$ we can write $W_t = \exp\{2\alpha/\sigma\, B_t\}$, so $\limsup_{t\to\infty} W_t = +\infty$, $\liminf_{t\to\infty} W_t = 0$. (The agent's wealth endlessly fluctuates between zero and infinity.) For $0 < p < \frac{1}{2}$ we get $W_t \xrightarrow{t\to\infty} 0$ a.s.

Proof Theorem 2 follows in the remainder of this appendix.

Lemma 1. *Let $\alpha > 0$, $M_t = \alpha\, t + B_t$, where $(B_t, t \geq 0)$ is the standard one-dimensional Brownian motion (and so M_t is a Brownian motion with drift α). Then $-\inf_{t>0} M_t$ is exponentially distributed with parameter 2α, i.e.,*

$$\mathrm{P}\left[\inf_{t>0} M_t \leq x\right] = \exp\{2\alpha x\}, \quad x \leq 0.$$

Proof. In [11, p. 197] the authors derive the probability $P(\alpha, d)$ that a Brownian motion with drift $\alpha > 0$ reaches a level $b < 0$ in finite time: $P(\alpha, d) = \exp\{2\alpha b\}$. The proof of Lemma 1 immediately follows since

$$P(\alpha, d) = \mathrm{P}\left[\inf_{t>0} M_t \leq x\right].$$

□

Proof of Theorem 2. By assumption, the stock price evolves as

$$dS_t = \alpha S_t \, dt + \sigma S_t \, dB_t, \quad \alpha \neq r, \; t \geq 0, ,$$

where B_t is the Brownian motion at time t. The discounted investor's wealth $\mathrm{e}^{-rt} W_t$ evolves as

$$d \, \mathrm{e}^{-rt} W_t = \pi_t \frac{\mathrm{e}^{-rt} W_t}{S_t} \, dS_t,$$

where $\pi_t \neq 0$ is the proportion of agent's wealth W_t invested in stock at time t. The stochastic process π_t satisfies the admissibility constraints.

Without loss of generality assume that $W_0 \equiv 1$ is the investor's initial wealth. The optimal proportion π_t is constant and is given by (7). Solving the preceding SDE yields, for any $p > 0$,

$$\mathrm{e}^{-rt} W_t = \exp\left\{ \left(\pi_p \alpha - \frac{1}{2} \pi_p^2 \sigma^2 \right) t + \pi_p \sigma \, B_t \right\} = \exp\left\{ \frac{1}{p} \left(1 - \frac{1}{2p} \right) \frac{\alpha^2}{\sigma^2} \, t + \frac{1}{p} \frac{\alpha}{\sigma} \, B_t \right\}. \tag{12}$$

Denote

$$M_t \triangleq p \frac{\sigma}{\alpha} \log \left(\mathrm{e}^{-rt} W_t \right) = B_t + \left(1 - \frac{1}{2p} \right) \frac{\alpha}{\sigma} \, t.$$

Then M_t is a Brownian motion with strictly positive drift. Using the result of Lemma 1 it follows

$$\mathrm{P}\left[\inf_{t \geq 0} M_t \leq y \right] = \exp\left\{ 2 \left(1 - \frac{1}{2p} \right) \frac{\alpha}{\sigma} \cdot y \right\}, \quad y \leq 0. \tag{13}$$

Since the function transforming W_t to M_t is strictly increasing, it follows that

$$\mathrm{P}\left[\inf_{t \geq 0} \left\{ \mathrm{e}^{-rt} W_t \right\} \leq x \right] = \mathrm{P}\left[\inf_{t \geq 0} M_t \leq p \frac{\sigma}{\alpha} \, \log x \right], \qquad x \in (0, 1]$$

which yields (11), after substituting $y = p \cdot \sigma / \alpha \cdot \log x$ into (13). □

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